Online Load Balancing and Machine Scheduling

– CT Network Theory Seminar –

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Outlines

1. Introduction of Online Balancing
2. Results of Greedy Algorithm
3. Constant Competitive Ratio for Related Machines
4. Competitive Ratio $O(\log n)$ for Unrelated Machines

Online Problems

- **Input:** *A request sequence* $\sigma = \sigma(1), \sigma(2), \ldots,;$
- **Online:** *An algorithm M* has to serve each request *online* without knowledge of future requests. Each request must be served before serving the next requests.
- **Objective:** The goal is to serve the entire request sequence so that the objective function is minimized.

The Online Load-Balancing Problem

Input:

- The n related/unrelated machines
- *A sequence of jobs* $\sigma = (\vec{p}(1), \vec{p}(2), \ldots, \vec{p}(k))$
- Each job $j \in \sigma$ denoted by its *"load vector"*:

$$\vec{p}(j) = (p_1(j), p_2(j), \ldots, p_n(j)), \text{ where } p_i(j) \geq 0.$$

Definition of Load:

- Assign a job j to a machine i increases the load on the machine i by $p_i(j)$
- $\ell_i(j)$ denotes the load on machine i after already assigning jobs 1 through j:

$$\ell_k(j) = \begin{cases} \ell_k(j-1) + p_k(j), & \text{if } k = i \\ \ell_k(j-1), & \text{otherwise} \end{cases}$$

The Online Load-Balancing Problem (2)

Notations of Offline Algorithm:

- Given *a sequence of jobs* $\sigma = (\vec{p}(1), \vec{p}(2), \ldots, \vec{p}(k))$
- $\mathscr{A}^*$ denotes off-line (optimal) algorithm
- $\ell_i^*(j)$ denotes the load on machine i achieved by $\mathscr{A}^*$ after assigning jobs 1 to j in σ

Objective:

- $L^*(k) = \max_i \ell_i^*(k)$: the maximum load (optimal) by $\mathscr{A}^*$
- $L(k) = \max_i \ell_i(k)$: the maximum load by online algorithms
- Online algorithms try to minimize $L(k)$
- Objective: minimize the competitive ratio:

  $$\sup\{L(k)/L^*(k)\} \text{ for any length } k \text{ and all possible sequences } \sigma$$

- k may be omitted for $L^*(k)$ and $L(k)$

Three Machine Models

Each job $j \in \sigma$ denoted by its *"load vector"*:

$$\vec{p}(j) = (p_1(j), p_2(j), \ldots, p_n(j))\,, \text{ where } p_i(j) \geq 0.$$

- Identical machimes: each job j: $p_i(j) = p_{i'}(j)$ for any two machines: i and i'
- Related machimes: for any jobs j and j' and machines i and i':

$$\frac{p_i(j)}{p_{i'}(j)} = \frac{p_i(j')}{p_{i'}(j')} = \frac{v_{i'}}{v_i}, \text{ where } v_i : \text{speed of machine } i$$

- Identical is a special case of related machines:
- Unrelated machimes $(1/\infty)$: all other cases
  - e.g., $p_i(j) = +\infty$, i.e., a job j not runnable on some machines i

Greedy Algorithm for Load Balancing Problem

Greedy Algorithm:

- Algorithm: for each job j, assign it to a machine k s.t., $k = \text{argmin}\{\ell_i(k-1) + p_i(j)\}$

Identical Machines:

- competitive ratio: $2 - \epsilon$

Related Machines:

- Competitive ratio: $O(\log n)$, lower bound: $\Omega(\log n)$

Unrelated Machines:

- Competitive ratio: $O(n)$, lower bound: $\Omega(n)$

Competitive Ratio of Greedy for Unrelated Machines

Theorem 2.1

The competitive ratio of the greedy algorithm is at most n for unrelated machines.

Proof.

- Each job j has a minimum load $\min_i p_i(j)$
- $\mathscr{S}_i^*$: jobs assigned to machine i by offline algorithms

$$nL^* \geq \sum_i \ell_i^* = \sum_i \sum_{j \in \mathscr{S}_i^*} p_i(j) \geq \sum_j \min_i p_i(j)$$

- Claim: the maximum load by greedy $\leq \sum_j \min_i p_i(j)$
- Assume: after assigning $j-1$, $L(j-1) \leq \sum_{j' \leq j-1} \min_i p_i(j')$
- When j arrives, greedy chooses machine m, the resulting load at most $L(j-1) + p_m(j) \leq \sum_{j' \leq j} \min_i p_i(j')$
- $L(j) \leq \sum_{j' \leq j} \min_i p_i(j')$ implies $L \leq \sum_j \min_i p_i(j') \leq nL^*$ $\square$

Algorithm Assign-R for Related Machines

procedure ASSIGN-R($\vec{p}, \vec{\ell}, \Lambda$);
  /* Λ — current estimate of L^*.
  Let $S := \{i | \ell_i + p_i \leq 2\Lambda\}$;
  if $S = \emptyset$
    then b :=fail
    else begin
      $k := \min\{i | i \in S\}$;
      $\ell_k := \ell_k + p_k$;
      b :=success
    end;
  return($\vec{\ell}, b$).
end.

FIG. 3. Algorithm ASSIGN-R.

Algorithm Assign-R for Related Machines

Theorem 3.1

If $\lambda^* \leq \Lambda$, then Algorithm ASSIGN-R *never fails. Thus, the load on a machine never exceeds* 2Λ

Proof.

- By contradiction, assumes it fails first on a job j.
- r: the fastest machine whose load $\leq L^*$, if no $r = 0$

$$r = \max\{i \mid \ell_i\,(j-1) \leq L^*\}$$

- Clearly, $r \neq n$, otherwise j is on n $(\ell_n(j-1) + p_n(j) \leq 2L^*)$
- Define $\Gamma = \{i \mid i > r\}$: overloaded machines faster than r
- Clearly, $\Gamma \neq \emptyset$ since $r < n$
- $\mathscr{S}_i/\mathscr{S}_i^*$: jobs assigned to machine i by online/offline algo.

Algorithm Assign-R for Related Machines

Proof (Cont.)

- Now, to show: it exists a job $s \in \bigcup_{i\in\Gamma} \mathscr{S}_i$ but $s \notin \bigcup_{i\in\Gamma} \mathscr{S}_i^*$

$$\sum_{i\in\Gamma, s\in\mathscr{S}_i} p_n(s) = \sum_{i\in\Gamma s, \in\mathscr{S}_i} \frac{p_n(s)}{p_i(s)} p_i(s)$$
$$= \sum_{i\in\Gamma} \frac{v_i}{v_n} \sum_{s\in\mathscr{S}_i} p_i(s) > \sum_{i\in\Gamma} \frac{v_i}{v_n} L^*$$
$$\geq \sum_{i\in\Gamma} \frac{v_i}{v_n} \sum_{s\in\mathscr{S}_i^*} p_i(s) = \sum_{i\in\Gamma, s\in\mathscr{S}_i^*} p_n(s)$$

- We know a job s assigned to $i \in \Gamma$ by online but to $i' \notin \Gamma$ by offline
- Our assumption: $p_{i'}(s) \leq L^* \leq \Lambda$
- Since $r \geq i'$, it implies r at least as fast as i'

Algorithm Assign-R for Related Machines

Theorem 3.1

If $\lambda^* \leq \Lambda$, then Algorithm ASSIGN-R *never fails. Thus, the load on a machine never exceeds* 2Λ

Proof (Cont.)

- We know a job s assigned to $i \in \Gamma$ by online but to $i' \notin \Gamma$ by offline
- Our assumption: $p_{i'}(s) \leq L^* \leq \Lambda$
- Since $r \geq i'$, it implies r at least as fast as i'
- Since job s assigned before j then $\ell_r(s-1) \leq \ell_r(j-1) \leq L^*$
- But, this means that online algo should have placed job s on r or a slower machine instead of i, which is a contradiction. □

Doubling Approach

Theorem 3.2

For any load balancing problem, let ALG_Λ be a parameterized online algorithm satisfying $\mathrm{OPT}(\sigma) \leq \Lambda \implies \mathrm{ALG}_\Lambda(\sigma) \leq c \cdot \Lambda$ (e.g., $\textsc{Assign-R}$ with $c = 2$ for related machines). Then there is an algorithm ALG s.t., for all σ, $\mathrm{ALG}(\sigma) \leq 4c \cdot \mathrm{OPT}(\sigma)$.

- ALG executes in stages, each stage correspond to the mos recent estimate of Λ.
- Stage 0, $\Lambda_0 = \mathrm{OPT}(j = 0)$, easy to compute the optimal for the first job.
- Each stage j, ALG uses ALG_Λ to assign jobs until it fails and start a new stage by doubling Λ
- Stage k, $\Lambda = 2^k \Lambda_0$

Proof of Doubling Approach

Proof.

- To prove $\mathrm{ALG}(\sigma) \leq 4c \cdot \mathrm{OPT}(\sigma)$ for any sequence σ
- Suppose ALG terminates at the stage h.
- If $h = 0$, it is clear $\mathrm{ALG}(\sigma) \leq c \cdot \mathrm{OPT}(\sigma)$
- Let r be the first job for the stage h, and σ_j denotes the subsequence processed in stage j
- Clearly, stage $h-1$ failed on $\sigma_{h-1}r$, while ALG_Λ has $\Lambda = 2^{h-1}\Lambda_0$
- Thus, $\mathrm{OPT}(\sigma) \geq \mathrm{OPT}(\sigma_{h-1}r) > 2^{h-1}\Lambda_0$
- Moreover
$$\mathrm{ALG}(\sigma) = \sum_{j=0}^{h} \mathrm{ALG}(\sigma_j) \leq \sum_{j=0}^{h} c \cdot 2^j\Lambda_0 = c\left(2^{h+1} - 1\right)\Lambda_0$$

$\square$

Competitive Ratio for Assign-R for Related Machines

Corollary 3.3

Assign-R *Algorithm can be modified to achieve a competitive ratio of* 8.

Algorithm Assign-U for Unelated Machines

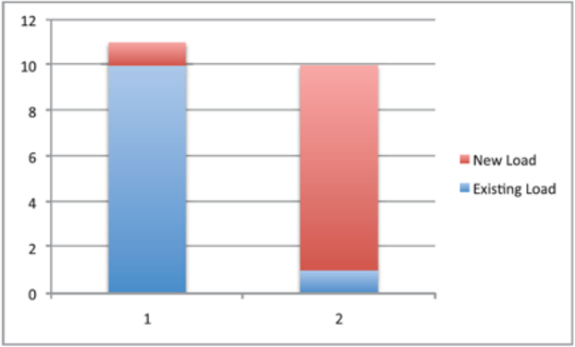
- Normalization: $\tilde{\ell}_i = \ell_i/\Lambda \ \tilde{p}_i = p_i/\Lambda$
- Find a machine s minimizing $\Delta_i = \alpha^{(\tilde{\ell}_i(j-1)+\tilde{p}_i(j))} - \alpha^{\tilde{\ell}_i(j-1)}$

procedure ASSIGN-U($\vec{p}, \vec{\ell}, \Lambda, \beta$);
  /* Λ — current estimate of L^*.
  /* β — designed performance guarantee of the algorithm.
  Let s be the index minimizing $\Delta_i = a^{(\bar{\ell}_i+\bar{p}_i)} - a^{\bar{\ell}_i}$;
  if $\ell_s + p_s > \beta\Lambda$
    then b :=fail
    else begin
      $\ell_s := \ell_s + p_s$;
      b :=success
    end;
  return($\vec{\ell}, b$).
end.

FIG. 2. Algorithm ASSIGN-U.

Intuitive Understanding of Assign-U Algorithm

- Assign-U minimizing $\alpha^{\tilde{\ell}_i(j-1)}\left(\alpha^{\tilde{p}_i(j)} - 1\right)$
- Greedy minimizing $\left(\tilde{\ell}_i\,(j-1) + \tilde{p}_i\,(j)\right)$
- Greedy choose machine 2
- But Assign-U chooses machine 1 since $1.5^{10+1} - 1.5^{10} \approx 22.83 < 51.16 \approx 1.5^{10} - 1.5^{1}$ $(\alpha = 1.5)$



Algorithm Assign-R for Related Machines

Theorem 4.1

If $\lambda^* \leq \Lambda$, then there exits $\beta = O(\log n)$ s.t., Algorithm $\textsc{Assign-U}$ *never fails. Thus, the load on a machine never exceeds $\beta\Lambda$*

Proof.

- State after $j-1$ jobs, $\alpha, \gamma > 1$ ($\alpha \approx 2, \gamma \approx 1$)
- assumption: $L^*(j) \leq L^* \leq \Lambda$
- potential function:

$$\Phi(j) = \sum_{i=1}^{n} \alpha^{\tilde{\ell}_i(j)} \left(\gamma - \tilde{\ell}_i^*(j)\right)$$

Algorithm Assign-R for Related Machines

Proof (Cont.)

- Let the job j assigned to machine i' (i) by online (offline)

$$\Phi(j) = \sum_{i=1}^{n} \alpha^{\tilde{\ell}_i(j)} \left(\gamma - \tilde{\ell}_i^*(j)\right)$$

- Compute $\Phi(j) - \Phi(j-1)$ as follows:

$$\begin{aligned}
&= \left(\gamma - \tilde{\ell}_{i'}^*(j-1)\right)\left(\alpha^{\tilde{\ell}_{i'}^*(j)} - \alpha^{\tilde{\ell}_{i'}^*(j-1)}\right) - \alpha^{\tilde{\ell}_i(j)}\tilde{p}_i(j) \\
&\le \gamma\left(\alpha^{\tilde{\ell}_{i'}^*(j-1)+\tilde{p}_{i'}(j)} - \alpha^{\tilde{\ell}_i(j-1)}\right) - \alpha^{\tilde{\ell}_i(j-1)}\tilde{p}_i(j) \\
&\le \gamma\left(\alpha^{\tilde{\ell}_i^*(j-1)+\tilde{p}_i(j)} - \alpha^{\tilde{\ell}_i(j-1)}\right) - \alpha^{\tilde{\ell}_i(j-1)}\tilde{p}_i(j) \\
&= \alpha^{\tilde{\ell}_i(j-1)}\left(\gamma\left(\alpha^{\tilde{p}_i(j)} - 1\right) - \tilde{p}_i(j)\right).
\end{aligned}$$

Algorithm Assign-R for Related Machines

Proof (Cont.)

- $\Phi(j) - \Phi(j-1) \leq \alpha^{\tilde{\ell}_i(j-1)} \left(\gamma \left(\alpha^{\tilde{p}_i(j)} - 1\right) - \tilde{p}_i(j)\right)$
- As offline assigns job j to machine i, $0 \leq \tilde{p}_i(j) \leq L^*/\Lambda \leq 1$
- To prove $\Phi(j) - \Phi(j-1) \leq 0$, we need to show $\gamma(\alpha^x - 1) \leq x$ for $x \in [0,1]$. It is true for $\alpha = 1 + 1/\gamma$
- Clearly, $\Phi(0) = \gamma n$
- Recall $\Phi(j) = \sum_{i=1}^{n} \alpha^{\tilde{\ell}_i(j)} \left(\gamma - \tilde{\ell}_i^*(j)\right)$
- Thus, $\sum_{i=1}^{n} \alpha^{\tilde{\ell}_i(j)} (\gamma - 1) \leq \gamma n$

$$\begin{aligned} L = \max_i \ell_i(k) &= \Lambda \max \tilde{\ell}_i(k) \\ &\leq \Lambda \cdot \log_a \left(\frac{\gamma}{\gamma - 1} n\right) = O(\Lambda \log n) \qquad \square \end{aligned}$$

Questions & Answers

Thanks for your Listening, and welcome your questions!